

A NEW SYMMETRY CRITERION BASED ON THE DISTANCE FUNCTION AND APPLICATIONS TO PDE'S

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ABSTRACT. We prove that, if $\Omega \subset \mathbb{R}^n$ is an open bounded starshaped domain of class C^2 , the constancy over $\partial\Omega$ of the function

$$\varphi(y) = \int_0^{\lambda(y)} \prod_{j=1}^{n-1} [1 - t\kappa_j(y)] dt$$

implies that Ω is a ball. Here $\kappa_j(y)$ and $\lambda(y)$ denote respectively the principal curvatures and the cut value of a boundary point $y \in \partial\Omega$. We apply this geometric result to different symmetry questions for PDE's: an overdetermined system of Monge-Kantorovich type equations (which can be viewed as the limit as $p \rightarrow +\infty$ of Serrin's symmetry problem for the p -Laplacian), and equations in divergence form whose solutions depend only on the distance from the boundary in some subset of their domain.

1. INTRODUCTION

Characterizing special classes of hypersurfaces in a metric space, in particular spheres, in terms of some properties of their principal curvatures, is a classical and challenging problem in Differential Geometry. A fundamental result by Alexandrov states that a bounded smooth domain in the Euclidean space is a ball provided the mean curvature of its boundary is constant [2]. For further characterizations of spheres involving the symmetric functions of the principal curvatures, see *e.g.* [41, 43] and the references therein.

Alexandrov's result has many powerful applications in Analysis, especially in the fields of PDE's and shape optimization; in fact, it allows to obtain for instance symmetry in overdetermined boundary value problems (as in the seminal paper [45] and the subsequent literature), of in extremum problems for variational functionals under geometric constraints (see the monograph [39]).

Often, symmetry questions arise in problems in which a crucial role is played by the distance function from the boundary of an open bounded domain $\Omega \subset \mathbb{R}^n$, $d_\Omega(x) := \text{dist}(x, \partial\Omega)$. This happens for instance when studying PDE's related with mass transportation theory (see [7, 13, 14]), or minimization problems in the class of so-called *web functions*, namely functions which only depend on d_Ω (see [19, 21, 22, 23, 24, 25, 37]). Symmetry questions in these frameworks, which will be described more precisely below, pushed us to set up a new roundedness criterion, which brings into play the distance function in a more intrinsic way than merely through the boundary curvatures. More precisely,

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it involves the following function φ associated with a bounded smooth domain Ω of \mathbb{R}^n :

$$(1) \quad \varphi(y) := \int_0^{\lambda(y)} \prod_{j=1}^{n-1} [1 - t\kappa_j(y)] dt, \quad y \in \partial\Omega.$$

Here $\kappa_j(\cdot)$ denote the principal curvatures of $\partial\Omega$, whereas $\lambda(y)$ is the *cut value* of y : letting $\nu(y)$ denote the unit outer normal to $\partial\Omega$ at y , and $\pi(x)$ be the point of $\partial\Omega$ such that $|x - \pi(x)| = \text{dist}(x, \partial\Omega)$ (which is uniquely determined for \mathcal{L}^n -a.e. $x \in \Omega$), the function $\lambda(\cdot)$ is defined on $\partial\Omega$ by

$$(2) \quad \lambda(y) := \sup \{t \geq 0 : \pi(y - t\nu(y)) = y\}, \quad y \in \partial\Omega.$$

Intuitively, by following the inner normal to $\partial\Omega$ at y , one can continue without crossing another straight line normal to $\partial\Omega$, exactly until one arrives at a distance $\lambda(y)$ from the boundary. Thus, Ω is filled up by the line segments $\{y - t\nu(y) : y \in \partial\Omega, t \in (0, \lambda(y))\}$.

Let us mention that the functions φ and λ already appeared in the literature in different contexts: concerning the function φ , it is a crucial tool in the proof of the isoperimetric inequality *à la Gromov* (see e.g. [3, §1.6.8]), and appears also in mathematical models for granular materials [14, 16, 26]; concerning the function λ , its regularity has been studied in [15, 40, 42], while some of its applications to variational problems can be found in [18, 21, 22, 23].

Our new symmetry criterion is formulated in terms of the function φ and of the mean curvature H of $\partial\Omega$,

$$(3) \quad H(y) := \frac{\kappa_1(y) + \cdots + \kappa_{n-1}(y)}{n-1}, \quad y \in \partial\Omega,$$

and reads as follows. By saying that Ω is *starshaped*, we mean that

$$(4) \quad \langle y, \nu(y) \rangle > 0 \quad \forall y \in \partial\Omega.$$

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set of class C^2 , starshaped with respect to the origin. Assume that there exists a point $y_0 \in \partial\Omega$ such that*

$$(5) \quad H(y_0) = \max_{y \in \partial\Omega} H(y), \quad \varphi(y_0) \geq \frac{|\Omega|}{|\partial\Omega|}.$$

Then Ω is a ball. In particular, if φ is constant on $\partial\Omega$, assumption (5) is satisfied, and hence Ω is a ball.

The proof of Theorem 1 is given in Section 3 below. It is obtained by showing that the existence of a point $y_0 \in \partial\Omega$ fitting the two conditions in (5) ensures the constancy of the mean curvature. To that aim, we exploit as crucial tools the arithmetic-geometric inequality and the Minkowski integral formula for the mean curvature (recalled in Section 2 below). In particular, in order to apply successfully such formula, we need the starshapedness condition (4): we believe it may be unnecessary for the validity of the result, but at present we are unable to circumvent it.

Concerning the C^2 -regularity assumption, in dimension $n = 2$ we are able to relax it, by allowing domains whose boundary is piecewise C^2 and may contain “convex” corners (see Section 3 for more details).

Let us now turn attention to describe two different applications of Theorem 1 to symmetry questions for PDE’s.

The first application concerns an overdetermined system of PDE’s of Monge-Kantorovich type, which can be viewed a limit version of Serrin’s symmetry result for the p -Laplacian

operator, as p tends to $+\infty$. To be more precise, let us recall Serrin's result: existence of a solution to the overdetermined boundary value problem

$$(6) \quad \begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u| = c & \text{on } \partial\Omega, \end{cases}$$

where c is a positive constant, implies that Ω is a ball. Later on, the same symmetry statement has been generalized to the case when the Laplacian in (6) is replaced by more general operators; since the literature on this topic is very broad, we limit ourselves to quote the paper [33], where the interested reader can also find many related references. In particular, Serrin's result extends to the system

$$(7) \quad \begin{cases} -\Delta_p u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u| = c & \text{on } \partial\Omega, \end{cases}$$

where Δ_p denotes the p -Laplacian operator (see [11, 28, 36]).

It is then natural to ask whether the same result continues to be true when considering, in some sense, the limit as $p \rightarrow +\infty$. Let us mention that, in this spirit, overdetermined boundary value problems for the ∞ -Laplacian have been recently studied in [12]. Here we look rather at the system of PDE's of Monge-Kantorovich type which arises as the limiting problem of (7) when $p \rightarrow +\infty$. Actually, let u_p be the unique solution to the Dirichlet boundary value problem given by the first two equations in (7). As explained more in detail in Section 4, passing to the limit as $p \rightarrow +\infty$ in such Dirichlet problem, leads to consider the following system in the unknowns $u \in \text{Lip}(\Omega)$ and $v \in L^1(\Omega; \mathbb{R}^+)$:

$$(8) \quad \begin{cases} -\text{div}(v\nabla u) = 1 & \text{in } \Omega, \\ |\nabla u| \leq 1 & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ (1 - |\nabla u|)v = 0 & \text{a.e. in } \Omega. \end{cases}$$

Heuristically, the variables u and v represent respectively the limit of u_p and $|\nabla u_p|^p$ as $p \rightarrow +\infty$ (cf. Section 4). System (8) always admits solutions; moreover, if (u, v) is a solution to (8), then $v \in C(\bar{\Omega})$ see [13, 14]). Therefore, the following question makes sense: does symmetry holds for (8) if, as a counterpart to the last equation in (7), we ask that v is constant on $\partial\Omega$? In other words:

$$(9) \quad \text{If (8) admits a solution } (u, v) \text{ with } v \text{ constant on } \partial\Omega, \text{ is } \Omega \text{ a ball?}$$

In Section 4, we show that the answer to question (9) is affirmative, as a consequence of Theorem 1. We provide two significant physical interpretations of this symmetry result in different frameworks: a shape optimization problem for heat conductors considered in [7], and a two-layers model in granular matter theory studied in [14].

The second application concerns "partially web solutions" to equations in divergence form. Let Ω be as above, and let ω be a bounded connected open subset of Ω ; we define the space of web functions on Ω and the space of their restrictions to ω respectively as

$$\begin{aligned} \mathcal{W}(\Omega) &:= \left\{ u \in W^{1,1}(\Omega) : u(x) \text{ depends only from } d_\Omega(x) \right\}, \\ \mathcal{W}(\Omega; \omega) &:= \left\{ u \in W^{1,1}(\Omega) : u(x) = \tilde{u}(x) \text{ a.e. on } \omega, \text{ for some } \tilde{u} \in \mathcal{W}(\Omega) \right\}. \end{aligned}$$

Given an equation in divergence form on Ω , with a constant source term,

$$(10) \quad -\operatorname{div}(A(|\nabla u|)\nabla u) = 1 \quad \text{in } \Omega ,$$

where $A \in C([0, +\infty))$, we say that u is a solution if $A(|\nabla u|)|\nabla u| \in L^1(\Omega)$ and

$$\int_{\Omega} A(|\nabla u|) \langle \nabla u, \nabla \psi \rangle dx = \int_{\Omega} \psi dx \quad \forall \psi \in C_0^\infty(\Omega) .$$

We then ask the following question:

$$(11) \quad \text{If (10) admits a solution } u \text{ belonging to } \mathcal{W}(\Omega; \omega), \text{ is } \Omega \text{ a ball?}$$

In order to provide some conditions on ω which are sufficient for a positive answer to question (11), we shall restrict attention to the case when ω is of the form

$$\Omega_\Gamma := \left\{ y - t\nu(y) : y \in \Gamma, t \in (0, \lambda(y)) \right\} ,$$

for some relatively open connected set $\Gamma \subseteq \partial\Omega$.

Let us remark that, if u is a solution to (10) belonging to $\mathcal{W}(\Omega; \Omega_\Gamma)$, since the restriction of u to Ω_Γ can be written as $h(d_\Omega)$ for some function h , it holds

$$u = h(0) \quad \text{and} \quad |\nabla u| = |h'(0)| \quad \text{on } \Gamma .$$

Hence, up to an additive constant, a solution u to (10) lying in $\mathcal{W}(\Omega; \Omega_\Gamma)$ satisfies the following system (which is clearly weaker than the requirement $u \in \mathcal{W}(\Omega; \Omega_\Gamma)$):

$$(12) \quad \begin{cases} -\operatorname{div}(A(|\nabla u|)\nabla u) = 1 & \text{in } \Omega , \\ u = 0 & \text{on } \Gamma , \\ |\nabla u| = c & \text{on } \Gamma . \end{cases}$$

In particular, if $\Gamma \equiv \partial\Omega$, the answer to question (11) is affirmative, as soon as the operator A is such that Serrin's symmetry result is valid for the elliptic equation (10); in this direction, let us mention that symmetry for minimizers to variational functionals in $\mathcal{W}(\Omega)$ was studied in [19].

Here we are rather interested in the case when Γ is strictly contained into $\partial\Omega$. We point out that system (12) is quite close to the the so-called partially overdetermined boundary value problems studied for the Laplace operator in [32]. However, in such partially overdetermined problems, one among the Dirichlet and the Neumann condition was required to hold on the whole of $\partial\Omega$ (see also [31, 34]), which is not necessarily the case when $u \in \mathcal{W}(\Omega; \Omega_\Gamma)$.

In Section 5, using Theorem 1, we obtain sufficient conditions on Γ for a positive answer to question (11) in dimension $n = 2$. We also compare more in detail our results and those obtained for partially overdetermined boundary value problems in [32].

The paper is organized as follows. After providing some preliminaries in Section 2, in Section 3 we give the proof of Theorem 1, and we generalize it to piecewise C^2 domains in the two-dimensional case. Sections 4 and 5 are devoted to the applications of the geometric result to PDE's, respectively to Monge-Kantorovich type equations, and to equations in divergence form with partially web solutions.

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2. NOTATION AND PRELIMINARIES

Let us fix some notation and geometric background.

The standard scalar product of two vectors $x, y \in \mathbb{R}^n$ is denoted by $\langle x, y \rangle$, and $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^n$. As is customary, $B_r(x_0)$ and $\overline{B}_r(x_0)$ are respectively the open and the closed ball centered at x_0 and with radius $r > 0$.

If Ω is an open subset of \mathbb{R}^n , we shall denote by $|\Omega|$ and $|\partial\Omega|$ respectively the n -dimensional Lebesgue measure of Ω and the $(n-1)$ -dimensional Hausdorff measure of its boundary.

A bounded open set $\Omega \subset \mathbb{R}^n$ (or, equivalently, its closure $\overline{\Omega}$ or its boundary $\partial\Omega$) is of class C^k , $k \in \mathbb{N}$, if for every point $x_0 \in \partial\Omega$ there exists a ball $B = B_r(x_0)$ and a one-to-one mapping $\psi: B \rightarrow D$ such that $\psi \in C^k(B)$, $\psi^{-1} \in C^k(D)$, $\psi(B \cap \Omega) \subseteq \{x \in \mathbb{R}^n; x_n > 0\}$, $\psi(B \cap \partial\Omega) \subseteq \{x \in \mathbb{R}^n; x_n = 0\}$.

Given a nonempty open set $\Omega \subset \mathbb{R}^n$, we denote by $d_\Omega: \overline{\Omega} \rightarrow \mathbb{R}$ the distance function from the boundary of Ω , defined by

$$d_\Omega(x) := \min_{y \in \partial\Omega} |x - y|, \quad x \in \overline{\Omega}.$$

It is well-known that d_Ω is a 1-Lipschitz function; by Rademacher theorem, its singular set Σ (i.e., the set of points where d_Ω is not differentiable) has zero Lebesgue measure (more precisely, it is a \mathcal{H}^{n-1} -rectifiable set, see [17, Prop. 4.1.3]). The closure of Σ is called the *cut locus* of Ω , and in general it may have a non-vanishing Lebesgue measure. Nevertheless, it can be shown that if Ω is of class C^2 , then $|\overline{\Sigma}| = 0$; moreover, in this case $\overline{\Sigma} \subset \Omega$ and d_Ω is of class C^2 in $\overline{\Omega} \setminus \overline{\Sigma}$ (see e.g. [26]).

Under the assumption that $\Omega \subset \mathbb{R}^n$ is a bounded open set of class C^2 , for every $y \in \partial\Omega$, we denote respectively by $\nu(y)$ and $T_y\Omega$ the unique outward unit normal vector and the tangent space of $\partial\Omega$ at y . The map $\nu: \partial\Omega \rightarrow S^{n-1}$ is called the *spherical image map* (or *Gauss map*). Then, for every $y \in \partial\Omega$, we denote by $\lambda(y)$ the *cut value* of y intended according to definition (2). It is well known that the singular set Σ is a subset of the collection C of all *cut points* $y - \lambda(y)\nu(y)$, $y \in \partial\Omega$; this set C has always vanishing Lebesgue measure (see [5]) and is contained in $\overline{\Sigma}$.

For every $y \in \partial\Omega$, the differential $d\nu_y$ of the Gauss map at y maps the tangent space $T_y\Omega$ into itself. The linear map $L_y := d\nu_y: T_y\Omega \rightarrow T_y\Omega$ is called the *Weingarten map*. The bilinear form defined on $T_y\Omega$ by $S_y(v, w) = \langle L_y v, w \rangle$, $v, w \in T_y\Omega$, is the *second fundamental form* of $\partial\Omega$ at y . The geometric meaning of the Weingarten map is the following: for every $v \in T_y\Omega$ with unit norm, $S_y(v, v)$ is equal to the normal curvature of $\partial\Omega$ at y in the direction v , that is, $S_y(v, v) = -\langle \dot{\xi}(0), \nu(y) \rangle$, where $\xi(t)$ is any parameterized curve in $\partial\Omega$ such that $\xi(0) = y$ and $\dot{\xi}(0) = v$. The eigenvalues $\kappa_1(y), \dots, \kappa_{n-1}(y)$ of the Weingarten map L_y are, by definition, the *principal curvatures* of $\partial\Omega$ at y . The corresponding eigenvectors are called the *principal directions* of $\partial\Omega$ at y . It is readily shown that every $\kappa_i(y)$ is the normal curvature of $\partial\Omega$ at y in the direction of the corresponding eigenvector. From the C^2 regularity assumption on the manifold $\partial\Omega$, it follows that the principal curvatures of $\partial\Omega$ are continuous functions on $\partial\Omega$. Their arithmetic mean is the mean curvature H of $\partial\Omega$, cf. definition (3). The following classical identity is usually referred as *Minkowski integral formula* for the mean curvature, see for instance [43, Section 2A]:

$$(13) \quad \int_{\partial\Omega} H(y) \langle y, \nu(y) \rangle d\mathcal{H}^{n-1} = |\partial\Omega|.$$

Finally, we shall need some elementary results about the relationship between cut value and boundary curvatures. In any space dimension, we have the following upper bound:

Lemma 2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set of class C^2 . Then, for every $y \in \partial\Omega$, it holds*

$$\kappa_i(y)\lambda(y) \leq 1 \quad \forall i = 1, \dots, n-1.$$

Proof. Given $y \in \partial\Omega$ let $x := y - \lambda(y)\nu(y)$ be its cut point. Since $d_\Omega(x) = \lambda(y)$, the open ball $B_{\lambda(y)}(x)$ is contained in Ω and is tangent to $\partial\Omega$ at y . Therefore, for every $i \in \{1, \dots, n-1\}$ we have either $\kappa_i(y) \leq 0$ or $1/\kappa_i(y) \geq \lambda(y)$. \square

In dimension $n = 2$, the cut value of boundary points with maximal curvature can be easily characterized as follows:

Lemma 3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set of class C^2 , and let $y_0 \in \partial\Omega$ be a maximum point of the curvature κ of $\partial\Omega$. Then*

$$\lambda(y_0) = 1/\kappa(y_0).$$

Proof. We observe that, since $\partial\Omega$ is compact, we have that $\kappa(y_0) > 0$. From Lemma 2, we know that $\kappa(y_0)\lambda(y_0) \leq 1$. Let us show the converse inequality. Setting $r := 1/\kappa(y_0)$, by assumption we have that $\kappa(y) \leq 1/r$ for every $y \in \partial\Omega$. By Schur's theorem for plane curves (see [30, Sect. 5-7, Ex.7]) we have that the disk $B = y_0 - r\nu(y_0) + B_r(0)$ is contained in Ω , so that $\lambda(y_0) \geq r$. \square

3. THE GEOMETRIC RESULT

This section is devoted to prove Theorem 1 and to extend it to less regular domains in the two-dimensional case.

Proof of Theorem 1. We divide the proof into four steps.

Step 1: upper bound for $\varphi(y)H(y)$.

As a first step, we claim that the following inequality holds true:

$$(14) \quad \varphi(y)H(y) \leq \frac{1}{n} \quad \forall y \in \partial\Omega.$$

Indeed, let us fix $y \in \partial\Omega$ and set $x_j := \lambda(y)\kappa_j(y)$, $j = 1, \dots, n-1$. Since $\lambda(y) > 0$, using the change of variables $s = t/\lambda(y)$ in the integral in (1) and multiplying by $H(y)$ we obtain the equality

$$(15) \quad \varphi(y)H(y) = \lambda(y)H(y) \int_0^1 \prod_{j=1}^{n-1} (1 - sx_j) ds = f(x_1, \dots, x_{n-1}),$$

where f is the function defined by

$$(16) \quad f(x_1, \dots, x_{n-1}) := \frac{x_1 + \dots + x_{n-1}}{n-1} \int_0^1 \prod_{j=1}^{n-1} (1 - sx_j) ds.$$

In view of (15) and (16), and since $x_j \leq 1$ by Lemma 2, in order to find an upper bound for the product $\varphi(y)H(y)$, we have to maximize f on the set

$$D := \{(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} : x_j \leq 1 \ \forall j\}.$$

Since $f(x_1, \dots, x_{n-1}) < 0$ if $x_1 + \dots + x_{n-1} < 0$, it is clear that $\max_D f$ is achieved on the compact set $K := D \cap \{x_1 + \dots + x_{n-1} \geq 0\}$. Let $(x_1, \dots, x_{n-1}) \in K$ and denote by \bar{x} their arithmetic mean. From the arithmetic–geometric mean inequality, taking into account that $\bar{x} \geq 0$, we have

$$\begin{aligned} f(x_1, \dots, x_{n-1}) &= \bar{x} \int_0^1 \prod_{j=1}^{n-1} (1 - s x_j) ds \\ &\leq \bar{x} \int_0^1 (1 - s \bar{x})^{n-1} ds \\ &= \frac{1}{n} [1 - (1 - \bar{x})^n] \leq \frac{1}{n}. \end{aligned}$$

Then the inequality (14) is proved.

Step 2: upper bound for $H(y)$.

Exploiting the assumptions (i) and (ii) made on the point y_0 , and applying (14) at $y = y_0$, we obtain

$$(17) \quad \frac{|\Omega|}{|\partial\Omega|} H(y) \leq \frac{|\Omega|}{|\partial\Omega|} H(y_0) \leq \varphi(y_0) H(y_0) \leq \frac{1}{n} \quad \forall y \in \partial\Omega.$$

Step 3: conclusion.

We multiply (17) by the positive quantity $\langle y, \nu(y) \rangle$: integrating on $\partial\Omega$, thanks to Minkowski integral formula (13) and the divergence theorem, we get

$$|\Omega| = \frac{|\Omega|}{|\partial\Omega|} \int_{\partial\Omega} H(y) \langle y, \nu(y) \rangle d\mathcal{H}^{n-1} \leq \frac{1}{n} \int_{\partial\Omega} \langle y, \nu(y) \rangle d\mathcal{H}^{n-1} = |\Omega|.$$

We deduce that

$$\frac{|\Omega|}{|\partial\Omega|} H(y) \langle y, \nu(y) \rangle = \frac{1}{n} \langle y, \nu(y) \rangle \quad \forall y \in \partial\Omega.$$

Recalling again that $\langle y, \nu(y) \rangle$ is a (strictly) positive quantity, we infer that the mean curvature of $\partial\Omega$ is constant, hence Ω is a ball by Alexandrov’s Theorem [2] (see also [44]).

Step 4: the case when φ is constant.

In order to obtain the second part of the statement, we recall the following change of variable’s formula which has been proved in [26, Theorem 7.1] for every $h \in L^1(\Omega)$:

$$(18) \quad \int_{\Omega} h(x) dx = \int_{\partial\Omega} \left[\int_0^{\lambda(y)} h(y - t\nu(y)) \prod_{j=1}^{n-1} [1 - t\kappa_j(y)] dt \right] d\mathcal{H}^{n-1}(y).$$

By applying it with $h \equiv 1$, we obtain that the mean value of φ over $\partial\Omega$ is precisely $|\Omega|/|\partial\Omega|$. Hence, if φ is constant on $\partial\Omega$, both conditions in (5) are fulfilled by choosing as y_0 any point of maximal mean curvature. \square

In the remaining of this section, we assume without any further mention that $n = 2$, and we study the validity of Theorem 1 for domains with piecewise smooth boundary. We start by noticing that, if “concave” corners are present in $\partial\Omega$, the change of variables formula (18) is no longer valid. In fact, in presence of such kind of corners, assuming that the function φ is constant in their complement, the conclusion of Theorem 1 does not remain true, as shown by the following

Example 4. Consider the set $\Omega = B_2(-1, 0) \cup B_2(1, 0) \subset \mathbb{R}^2$. In this case Ω has two “concave” corners at $\mathcal{C} = \{(0, \sqrt{3}), (0, -\sqrt{3})\}$, $\varphi(y) = 1$ for every $y \in \partial\Omega \setminus \mathcal{C}$, but Ω is not a ball.

In view of the above Example, we are going to restrict our attention to the following class of domains.

Definition 5. We say that a bounded connected open set $\Omega \subset \mathbb{R}^2$ is *piecewise C^2 without concave corners* if it satisfies a uniform exterior sphere condition, and

$$\partial\Omega = \bigcup_{i=1}^m \Gamma_i ,$$

where each Γ_i is a simple C^2 arc up to its endpoints y_{i-1} and y_i (with the convention $y_0 = y_m$), and

$$\Gamma_i \cap \Gamma_j = \begin{cases} \{y_i\}, & \text{if } 1 \leq i \leq m-1, j = i+1, \\ \{y_m\}, & \text{if } i = m, j = 1, \\ \emptyset, & \text{if } i - j \neq \pm 1. \end{cases}$$

Clearly, the boundary of a domain Ω as in Definition 5 may contain “convex” corners at points in the set

$$\mathcal{C} := \{y_1, \dots, y_m\} .$$

We point out that, for such domains, the change of variables formula (18) still holds (see [20, Thm. 3.3]), whereas the Minkowski integral formula (13) becomes

$$(19) \quad |\partial\Omega| = \int_{\partial\Omega} \kappa(y) \langle y, \nu(y) \rangle d\mathcal{H}^1(y) - \sum_{i=1}^m \langle y_i, R\Delta\nu(y_i) \rangle ,$$

where $R := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\Delta\nu(y_i) = \nu_{i+1}(y_i) - \nu_i(y_i)$ (being ν_i the outer normal to the boundary component Γ_i), and

$$\nu_i(y_j) := \lim_{\substack{y \rightarrow y_j \\ y \in \Gamma_i}} \nu_i(y)$$

(see [1]).

Consequently, the following generalized versions of Theorem 1 hold.

Theorem 6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set piecewise C^2 without concave corners, and starshaped with respect to the origin. Assume that one of the following conditions is satisfied:*

- (i) Ω is of class C^1 and there exists a point $y_0 \in \partial\Omega \setminus \mathcal{C}$ such that

$$H(y_0) = \max_{y \in \partial\Omega \setminus \mathcal{C}} H(y) , \quad \varphi(y_0) \geq \frac{|\Omega|}{|\partial\Omega|} .$$

- (ii) φ is constant in $\partial\Omega \setminus \mathcal{C}$.

Then Ω is a ball.

Proof. Assume first that Ω satisfies condition (i). In this case, all the steps of the proof of Theorem 1 can be repeated. In particular, Step 3 continues to hold thanks to the hypothesis that Ω is of class C^1 : indeed, such assumption ensures that each addendum of the finite sum in the r.h.s. of (19) vanishes, so that the Minkowski formula is still valid for Ω in the very same form as if it was a C^2 domain.

Assume now that Ω satisfies condition (ii). Similarly as in Step 4 of the proof of Theorem 1, since φ is constant, from the change of variable's formula (18) we get that $\varphi(y) = |\Omega|/|\partial\Omega|$ for every $y \in \partial\Omega \setminus \mathcal{C}$.

This fact has two consequences.

As a first consequence, the upper bound inequality (17) in Step 2 holds in the form

$$(20) \quad \frac{|\Omega|}{|\partial\Omega|} \kappa(y) \leq \frac{1}{2}, \quad \forall y \in \partial\Omega \setminus \mathcal{C}.$$

Namely, if $\kappa(y) \leq 0$ the inequality is trivial, whereas, if $\kappa(y) > 0$, then $\lambda(y) \leq 1/\kappa(y)$ and

$$\frac{|\Omega|}{|\partial\Omega|} = \varphi(y) = \lambda(y) - \frac{\lambda(y)^2}{2} \kappa(y) \leq \frac{1}{2\kappa(y)}$$

and (20) follows.

As a second consequence, $\partial\Omega$ cannot have (convex) corners. Namely, if y_i is a point with $\langle \nu_i(y_i), \nu_{i+1}(y_i) \rangle < 1$, a simple geometric argument shows that $\lim_{y \rightarrow y_i} \lambda(y) = 0$. Since κ is uniformly bounded on $\partial\Omega \setminus \mathcal{C}$, we deduce that $\lim_{y \rightarrow y_i} \varphi(y) = 0$ and, by assumption, we conclude that $\varphi = 0$ on $\partial\Omega \setminus \mathcal{C}$, a contradiction.

Now, since (20) holds and the absence of convex corners ensures that the Minkowski formula is satisfied for Ω in the very same form as if it was a C^2 domain, the conclusion of the proof is identical to Step 3 in the proof of Theorem 1. \square

4. APPLICATION TO PDE'S OF MONGE-KANTOROVICH TYPE

In this section we deal with question (9) stated in the Introduction. We begin by explaining more in detail its relationship with the overdetermined problem (7) for the p -Laplacian. Let $\Omega \subset \mathbb{R}^n$ be an open bounded connected set of class C^2 , and let f be a signed measure with finite total variation supported in $\overline{\Omega}$. For every $p > 1$, set

$$\alpha_p := \min \left\{ \int_{\Omega} \frac{1}{p} |\nabla u|^p dx - \langle f, u \rangle : u \in W_0^{1,p}(\Omega) \right\},$$

and denote by u_p the unique minimizer, namely the unique solution to the boundary value problem

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It was proved in [8] that, as $p \rightarrow +\infty$,

$$(21) \quad \alpha_p \rightarrow \alpha$$

$$(22) \quad u_p \rightarrow u \quad \text{uniformly}$$

$$(23) \quad |\nabla u_p|^{p-2} \rightharpoonup \mu \quad \text{weakly as measures,}$$

being

$$(24) \quad \alpha := -\sup \left\{ \langle f, u \rangle : u \in C_0^\infty(\mathbb{R}^n), |\nabla u| \leq 1 \text{ on } \Omega, u = 0 \text{ on } \partial\Omega \right\}$$

and (u, μ) a solution to

$$(25) \quad \begin{cases} -\operatorname{div}((D_\mu u)\mu) = f & \text{in } \Omega, \\ u \in \operatorname{Lip}_1(\Omega, \partial\Omega), \\ |D_\mu u| = 1 & \mu\text{-a.e.} \end{cases}$$

Here

$$\text{Lip}_1(\Omega, \partial\Omega) = \left\{ u \in W^{1,\infty}(\Omega) : u = 0 \text{ on } \partial\Omega \text{ and } |\nabla u| \leq 1 \text{ a.e. in } \Omega \right\},$$

where, thanks to the regularity of Ω , we understand that a function $u \in W^{1,\infty}(\Omega)$ is continuously extended to $\overline{\Omega}$, while μ is a positive measure supported on $\overline{\Omega}$, and $D_\mu u$ denotes the μ -tangential gradient of u , which can be suitably defined starting from the notion of tangent space to μ introduced in [9] (see also [35]).

Here we skip to introduce the tangential calculus with respect to a measure (for which we refer *e.g.* to the survey paper [10]), since we are going to restrict our attention to the case when μ is an absolutely continuous measure:

$$\mu = v(x) dx, \text{ with } v \in L^1(\Omega), \quad v \geq 0.$$

In this case, the μ -tangential gradient of u agrees with the gradient of u on the set $\{v > 0\}$, and vanishes on the complementary set $\{v = 0\}$. Thus system (25) can be rewritten as

$$(26) \quad \begin{cases} -\text{div}(v \nabla u) = f & \text{in } \Omega, \\ u \in \text{Lip}_1(\Omega, \partial\Omega), \\ (1 - |\nabla u|)v = 0 & \text{a.e. in } \Omega. \end{cases}$$

In view of the convergence property (23), we paraphrase the classical overdetermined Neumann condition for system (7), namely

$$|\nabla u_p| = c \quad \text{on } \partial\Omega,$$

with the the following overdetermined condition for system (25):

$$(27) \quad \mu = v(x) dx, \text{ with } v \in L^1(\Omega), \quad v \geq 0, \quad v = c \text{ on } \partial\Omega.$$

By this way, we arrive at question (9): under the assumption that the source f is a positive constant on $\overline{\Omega}$, does the existence of a solution to (25)-(27) imply the roundness of Ω ?

Remark 7. We point out that the constancy condition asked for v on the boundary in (27) is meaningful as soon as the source f is assumed to be a continuous function on $\overline{\Omega}$, because in this case the v -component of a solution (u, v) to (26) is itself a continuous function (see [14, Prop. 3.2]). Moreover, we recall that the v -component is unique (see [14, Thm. 4.1], [27, Section 6] and the proof of Theorem 8 below), whereas the u -component is unique (and coincides with d_Ω) if and only if the singular set Σ of d_Ω is contained in the support of the source f [27, Section 7]. In particular, if f is a positive constant, then $u = d_\Omega$ and u_p converges uniformly to d_Ω as $p \rightarrow \infty$ (see also [4] and [12, Remark 2.1]).

The next result gives an affirmative answer to question (9) (under the appropriate assumptions on Ω which allow to apply Theorem 1; a similar statement clearly holds in the nonsmooth two-dimensional case covered by Theorem 6).

Theorem 8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set of class C^2 , starshaped with respect to the origin. If the source f is positive and constant on $\overline{\Omega}$ and problem (25)-(27) admits a solution, then Ω is a ball.*

Proof. Since $f \in C(\overline{\Omega})$, one solution to problem (26) is given by the pair (d_Ω, v_f) , where $v_f: \overline{\Omega} \rightarrow \mathbb{R}$ is the (continuous) function defined by

$$(28) \quad v_f(x) = \begin{cases} \int_0^{\tau(x)} f(x + t\nu(x)) \prod_{i=1}^{n-1} \frac{1 - (d_\Omega(x) + t)\kappa_i(x)}{1 - d_\Omega(x)\kappa_i(x)} dt & \text{if } x \in \overline{\Omega} \setminus \overline{\Sigma}, \\ 0, & \text{if } x \in \overline{\Sigma} \end{cases}$$

(see [14, Thm. 3.1]). Here, for every $x \in \overline{\Omega} \setminus \overline{\Sigma}$, we have used the notation $\kappa_i(x) := \kappa_i(\pi(x))$, $\nu(x) := \nu(\pi(x))$, and $\tau(x) := \lambda(\pi(x)) - d_\Omega(x)$. Moreover, the following uniqueness result holds for the v -component of the system: if (u, v) is any solution to (26), then $v = v_f$ (see [14, Thm. 4.1]).

In particular, in the case $x \in \partial\Omega$ (i.e. $d_\Omega(x) = 0$), the representation formula (28) can be rewritten as:

$$v_f(x) = \int_0^{\lambda(x)} f(x + t\nu(x)) \prod_{i=1}^{n-1} (1 - t\kappa_i(x)) dt, \quad x \in \partial\Omega.$$

From such representation formula at the boundary and the uniqueness of the v -component, when f is a positive constant γ for every $x \in \overline{\Omega}$, we have

$$v(y) = v_f(y) = \gamma\varphi(y) \quad \forall y \in \partial\Omega,$$

where φ is precisely the function defined at (1). The conclusion now follows from Theorem 1. \square

Theorem 8 admits the following physical interpretations.

– *A thermic model.* Consider the problem of finding a positive measure μ which solves the variational problem

$$(29) \quad \min \left\{ \mathcal{C}(\mu) : \int d\mu = m, \text{ spt}(\mu) \subseteq \overline{\Omega} \right\},$$

where m is a prescribed positive parameter, and the cost \mathcal{C} is given by

$$\mathcal{C}(\mu) := - \inf \left\{ \int j(\nabla u) d\mu - \langle f, u \rangle : u \in \mathcal{D}(\mathbb{R}^n), u = 0 \text{ on } \partial\Omega \right\},$$

being j a stored energy density (typically, $j(z) = \frac{1}{2}|z|^2$), and f a signed measure with finite total variation supported on $\overline{\Omega}$. If μ is interpreted as the distribution of a conducting material, u as the temperature (which is kept 0 at the boundary), and f as a given heat sources density, (29) can be interpreted as the optimal design problem of finding the most performant conductor of prescribed mass in the design region $\overline{\Omega}$.

A key result established in [7, Theorem 2.3] states that the minimum in (29) is equal to

$$\frac{\alpha^2}{2m},$$

with α as in (21). Moreover, up to constant multiples, system (25) provides necessary and sufficient optimality conditions on u and μ for being a solution respectively to (24) and (29) [7, Theorem 3.9].

In the light of these results, Theorem 8 can be interpreted as follows:

Assume that a constant source heats a region $\overline{\Omega}$, and that the most performant conductor in such region is represented by a function constant on the boundary $\partial\Omega$. Then Ω is a ball.

– *A model for sandpiles.* In the dynamical theory of granular matter the so-called table problem consists in studying the evolution of a sandpile created by pouring dry matter (the sand) onto a table, which is represented by a bounded open set $\Omega \subset \mathbb{R}^2$, while the (time-independent) matter source is represented by a non-negative function $f \in L^1(\Omega)$. Among differential models, in the so-called BCRE model (after Bouchaud, Cates, Ravi Prakash, Edwards [6]) and its successive modifications (see [38]), the description of the growing sandpile is based on the introduction of two layers, the *standing layer*, which collects the matter that remains at rest, and the *rolling layer*, which is the thin layer of matter moving down along the surface of the standing layer.

The equilibrium solutions of this model are precisely the solutions to (26), where u and v denote respectively the thickness of the standing and rolling layer.

Theorem 8 has then the following straightforward meaning:

Assume that a uniformly distributed (and constant in time) amount of dry sand is poured onto a table $\Omega \subset \mathbb{R}^2$, and that, once the equilibrium is reached, the thickness of the rolling layer is constant on the boundary $\partial\Omega$. Then Ω is a disk.

5. APPLICATION TO PDE'S WITH PARTIALLY WEB SOLUTIONS

In this section we deal with question (11) stated in the Introduction. We adopt the same notation for the set Ω_Γ and the space $\mathcal{W}(\Omega; \Omega_\Gamma)$.

The next result gives sufficient conditions on Γ for an affirmative answer in space dimensions $n = 2$; κ denotes the curvature of $\partial\Omega$.

Theorem 9. *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set of class C^2 , starshaped with respect to the origin. Assume that $A \in C([0, +\infty))$ and that there exists a solution u to equation (10) in Ω belonging to the space*

$$(30) \quad \mathcal{W}^1(\Omega; \Omega_\Gamma) := \left\{ u \in \mathcal{W}(\Omega; \Omega_\Gamma) : u|_{\overline{\Omega_\Gamma}} \in C^1 \right\},$$

where Γ is a relatively open connected subset of $\partial\Omega$ such that

- (i) the maximum of κ on $\partial\Omega$ is attained on Γ ;
- (ii) for every $\epsilon > 0$ small enough, a.e. in $\Omega_\epsilon := \{x \in \Omega : d_\Omega(x) < \epsilon\}$ there holds:

$$A(|\nabla u|) \langle \nabla u, \nabla d_\Omega \rangle \leq [-A(|\nabla u|)u_\nu]|_\Gamma + o(1)$$

(where $o(1)$ denotes a quantity that vanishes as $\epsilon \rightarrow 0$).

Then Ω is a ball.

Remark 10. In assumption (ii), $[-A(|\nabla u|)u_\nu]|_\Gamma$ denotes the restriction of $[-A(|\nabla u|)u_\nu]$ to Γ , which is constant since $u \in \mathcal{W}(\Omega; \Omega_\Gamma)$. We point out that, if one assumes further that u is of class $C^1(\overline{\Omega_\epsilon})$, condition (ii) can be rewritten in a more comfortable way. Namely, in this case it becomes

- (ii') the maximum of $(-A(|\nabla u|)u_\nu)$ on $\partial\Omega$ is attained on Γ .

Theorem 9 should be compared with the results obtained for partially overdetermined boundary value problems obtained in [32]. Therein, it was proved in particular that existence of a solution to the boundary value problem

$$(31) \quad \begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u| = c & \text{on } \Gamma, \end{cases}$$

implies that Ω is a ball under one of the following assumptions:

- $\partial\Omega$ is connected, and $\Gamma \subseteq \partial\tilde{\Omega}$ for some open set $\tilde{\Omega}$ with connected analytic boundary (cf. [32, Theorem 1]) ;
- $\partial\Omega \in C^{2,\alpha}$, $\sup_{x \in \Gamma} H(x) \geq 1/nc$ and the maximum of $|\nabla u|$ over $\partial\Omega$ is attained on Γ (cf. [32, Theorem 3]).

In comparison with [32, Theorem 1], let us remark that the proof of Theorem 9 is straightforward in case A is the Laplacian or any other operator in divergence form for which one knows the following two facts: equation (10) has a analytic solution inside Ω and Serrin's symmetry result holds true. Indeed in this case, by uniqueness of the analytic continuation, the solution which is assumed to exist in $\mathcal{W}(\Omega; \Omega_\Gamma)$ agrees with the analytic solution to (10) on whole of Ω , so that u satisfies at the same time a constant Dirichlet and Neumann condition on the entire $\partial\Omega$, and the conclusion follows from Serrin's result. Clearly, this kind of argument does not apply any longer when dealing with degenerated elliptic operators such as the p -Laplacian, for which analytic regularity of solutions on the whole of Ω is not fulfilled (see *e.g.* [29, 46]).

On the other hand, it is interesting to observe that the assumptions of Theorem 9 are quite similar to those of [32, Theorem 3], though (as already mentioned in the Introduction) neither the existence of a solution $u \in \mathcal{W}(\Omega; \Omega_\Gamma)$ implies the existence of a solution to (31) nor the converse, and though the proof techniques are completely different.

The proof of Theorem 9 relies on Theorem 1 combined with Proposition 11 below, where we establish a link holding, at a point of maximal curvature, between the normal derivative of partially web solutions to (10), and the function φ introduced in (1).

Proposition 11. *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set of class C^2 , starshaped with respect to the origin. Assume there exists a solution u to equation (10) in Ω belonging to the space $\mathcal{W}^1(\Omega; \Omega_\Gamma)$ defined in (30), where Γ is a relatively open connected subset of $\partial\Omega$ such that the maximum of κ on $\partial\Omega$ is attained at some point $y_0 \in \Gamma$.*

Then the following identity holds at y_0 :

$$A(|\nabla u|(y_0))u_\nu(y_0) = -\varphi(y_0) .$$

Proof. Let $Y: (-r, r) \rightarrow \partial\Omega$ be a local parametrization of Γ by arc-length, such that $Y(0) = y_0$. For $\rho \in (-r, r)$, let

$$\Gamma_\rho := Y(-\rho, \rho) .$$

For ρ as above and $t \in [0, \Lambda(\rho)]$, we set

$$\Lambda(\rho) := \lambda(Y(\rho)) , \quad K(\rho) := \kappa(Y(\rho)) ,$$

and

$$T(\rho) := \tau(Y(\rho)) = \dot{Y}(\rho) , \quad N(\rho) := \nu(Y(\rho)) , \quad X(\rho, t) = Y(\rho) - tN(\rho) .$$

By assumption, (10) admits a solution $u \in \mathcal{W}^1(\Omega; \Omega_\Gamma)$, so we can write $u(x) = h(d_\Omega(x))$ for every $x \in \bar{\Omega}_\Gamma$, with $h \in C^1$, and it holds

$$(32) \quad \nabla u(X(\rho, t)) = -h'(t)N(\rho) .$$

We now construct a suitable family of test function to be used in equation (10).

Let $\Lambda_\rho := \min_{|\sigma| \leq \rho} \Lambda(\sigma)$. For $\epsilon > 0$ small enough let $\phi_\epsilon, \eta_\epsilon: \mathbb{R} \rightarrow \mathbb{R}$ be the functions defined by

$$\phi_\epsilon(t) := \begin{cases} 0 & \text{if } t \leq 0 \text{ or } t \geq \Lambda_\rho - \epsilon, \\ 1 & \text{if } t \in [\epsilon, \Lambda_\rho - 2\epsilon], \\ \frac{t}{\epsilon} & \text{if } t \in (0, \epsilon), \\ \frac{\Lambda_\rho - \epsilon - t}{\epsilon} & \text{if } t \in (\Lambda_\rho - 2\epsilon, \Lambda_\rho - \epsilon), \end{cases} \quad \eta_\epsilon(\sigma) := \begin{cases} 0 & \text{if } |\sigma| \geq \rho, \\ 1 & \text{if } |\sigma| \leq \rho - \epsilon, \\ \frac{\rho - |\sigma|}{\epsilon} & \text{if } \rho - \epsilon < |\sigma| < \rho. \end{cases}$$

Then, for $\rho \in (-r, r)$ and ϵ small enough, we consider the family of functions $\psi_{\rho, \epsilon}: \Omega \rightarrow \mathbb{R}$ given by

$$\psi_{\rho, \epsilon}(x) := \begin{cases} \phi_\epsilon(t)\eta_\epsilon(\sigma), & \text{if } x = X(\sigma, t) \text{ for some } (\sigma, t) \in D_\rho := (-\rho, \rho) \times (0, \Lambda_\rho - \epsilon), \\ 0, & \text{otherwise.} \end{cases}$$

Since X is a C^1 diffeomorphism from $\overline{D_\rho}$ to $X(\overline{D_\rho})$, each function $\psi_{\rho, \epsilon}$ is Lipschitz continuous. Therefore, it can be taken as a test function in equation (10). Passing to the limit as $\epsilon \rightarrow 0$, then dividing by $|\Gamma_\rho|$ and passing to the limit also as $\rho \rightarrow 0$, we obtain

$$(33) \quad \lim_{\rho \rightarrow 0} \left\{ \frac{1}{|\Gamma_\rho|} \lim_{\epsilon \rightarrow 0} \int_{\Omega} \langle A(|\nabla u|) \nabla u, \nabla \psi_{\rho, \epsilon} \rangle dx \right\} = \lim_{\rho \rightarrow 0} \left\{ \frac{1}{|\Gamma_\rho|} \lim_{\epsilon \rightarrow 0} \int_{\Omega} \psi_{\rho, \epsilon} dx \right\}.$$

The right hand side of (33) is immediately computed as

$$(34) \quad \lim_{\rho \rightarrow 0} \left\{ \frac{1}{|\Gamma_\rho|} \lim_{\epsilon \rightarrow 0} \int_{\Omega} \psi_{\rho, \epsilon} dx \right\} = \lim_{\rho \rightarrow 0} \frac{|\Omega_\rho|}{|\Gamma_\rho|} = \lim_{\rho \rightarrow 0} \frac{\int_{\Gamma_\rho} \varphi d\mathcal{H}^1}{|\Gamma_\rho|} = \varphi(y_0).$$

Let us compute the left hand side of (33).

Differentiating the relation $\psi_{\rho, \epsilon}(X(\sigma, t)) = \phi_\epsilon(t)\eta_\epsilon(\sigma)$ with respect to t we get

$$(35) \quad \langle \nabla \psi_{\rho, \epsilon}(X(\sigma, t)), N(\sigma) \rangle = -\phi'_\epsilon(t)\eta_\epsilon(\sigma) \quad \text{a.e. on } D_\rho.$$

By combining (32) and (35), we infer

$$\begin{aligned} \langle A(|\nabla u|) \nabla u, \nabla \psi_{\rho, \epsilon} \rangle(X(\sigma, t)) &= -A(|h'(t)|)h'(t) \langle N(\sigma), \nabla \psi_{\rho, \epsilon}(X(\sigma, t)) \rangle \\ &= A(|h'(t)|)h'(t)\phi'_\epsilon(t)\eta_\epsilon(\sigma) \quad \text{a.e. on } D_\rho. \end{aligned}$$

Then, by using the change of variables formula (18), we get

$$\begin{aligned} &\int_{\Omega} \langle A(|\nabla u|) \nabla u, \nabla \psi_{\rho, \epsilon} \rangle dx \\ &= \int_{D_\rho} \langle A(|\nabla u|) \nabla u, \nabla \psi_{\rho, \epsilon} \rangle dx \\ &= \int_{-\rho}^{\rho} \int_0^{\Lambda(\sigma)} A(|h'(t)|)h'(t)\phi'_\epsilon(t)\eta_\epsilon(\sigma) [1 - tK(\sigma)] dt d\sigma \\ &= \frac{1}{\epsilon} \int_{-\rho}^{\rho} \int_0^{\epsilon} A(|h'(t)|)h'(t)\eta_\epsilon(\sigma) [1 - tK(\sigma)] dt d\sigma \\ &\quad - \frac{1}{\epsilon} \int_{-\rho}^{\rho} \int_{\Lambda_\rho - 2\epsilon}^{\Lambda_\rho - \epsilon} A(|h'(t)|)h'(t)\eta_\epsilon(\sigma) [1 - tK(\sigma)] dt d\sigma. \end{aligned}$$

In the limit as $\epsilon \rightarrow 0^+$, by the regularity assumption made on the solution u and on the operator A (recall that $h \in C^1$ and $A \in C([0, +\infty))$), we obtain

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0^+} \int_{\Omega} \langle A(|\nabla u|) \nabla u, \nabla \psi_{\rho, \epsilon} \rangle dx \\ &= \int_{-\rho}^{\rho} \left\{ A(|h'(0)|) h'(0) - A(|h'(\Lambda_{\rho})|) h'(\Lambda_{\rho}) [1 - \Lambda_{\rho} K(\Lambda_{\rho})] \right\} d\sigma \\ &= |\Gamma_{\rho}| \left\{ A(|h'(0)|) h'(0) - A(|h'(\Lambda_{\rho})|) h'(\Lambda_{\rho}) [1 - \Lambda_{\rho} K(\Lambda_{\rho})] \right\}. \end{aligned}$$

In the limit as $\rho \rightarrow 0^+$, by exploiting again the regularity assumptions recalled above, we obtain

$$\lim_{\rho \rightarrow 0^+} A(|h'(\Lambda_{\rho})|) h'(\Lambda_{\rho}) [1 - \Lambda_{\rho} K(\Lambda_{\rho})] = A(|h'(\lambda(y_0))|) h'(\lambda(y_0)) [1 - \lambda(y_0) \kappa(y_0)] = 0,$$

where the last equality follows from Lemma 3.

We have thus proved that the left hand side of (33) is given by

$$(36) \quad \lim_{\rho \rightarrow 0} \left\{ \frac{1}{|\Gamma_{\rho}|} \lim_{\epsilon \rightarrow 0} \int_{\Omega} \langle A(|\nabla u|) \nabla u, \nabla \psi_{\rho, \epsilon} \rangle dx \right\} = A(|h'(0)|) h'(0) = -A(|\nabla u|(y_0)) u_{\nu}(y_0).$$

The proof is achieved by combining (33), (34), and (36). \square

Proof of Theorem 9. By assumption (i), there exists a point $y_0 \in \Gamma$ such that

$$(37) \quad \kappa(y_0) = \max_{y \in \partial\Omega} \kappa(y).$$

By Proposition 11, it holds

$$\varphi(y_0) = -A(|\nabla u|(y_0)) u_{\nu}(y_0).$$

We now exploit again the assumption that $u = h(d_{\Omega})$ in $\overline{\Omega_{\Gamma}}$, for some function $h : \mathbb{R}^+ \rightarrow \mathbb{R}$ of class C^1 . Thus $A(|\nabla u|) u_{\nu}$ is constant on Γ , so that

$$\varphi(y_0) = -A(|h'(0)|) h'(0) = [-A(|\nabla u|) u_{\nu}]|_{\Gamma}$$

and assumption (ii) becomes

$$A(|\nabla u|) \langle \nabla u, \nabla d_{\Omega} \rangle \leq \varphi(y_0) + o(1) \quad \text{a.e. in } \Omega_{\epsilon}.$$

Using $\psi_{\epsilon}(x) := \min\{\epsilon^{-1} d_{\Omega}(x), 1\}$ as a test function in (10) we obtain, for $\epsilon > 0$ small enough,

$$\int_{\Omega} \psi_{\epsilon} dx = \frac{1}{\epsilon} \int_{\Omega_{\epsilon}} A(|\nabla u|) \langle \nabla u, \nabla d_{\Omega} \rangle dx \leq \varphi(y_0) |\partial\Omega| + o(1)$$

and, taking the limit as $\epsilon \rightarrow 0$,

$$|\Omega| \leq \varphi(y_0) |\partial\Omega|.$$

This inequality, together with (37), ensures that the hypotheses of Theorem 1 are satisfied. Hence Ω must be a ball. \square

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